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Remarks about Disjoint Dominating Sets

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Abstract

We solve a number of problems posed by Hedetniemi, Hedetniemi, Laskar, Markus, and Slater concerning pairs of disjoint sets in graphs which are dominating or independent and dominating.

Keywords: domination; independence; inverse domination
AMS subject classification: 05C69

1 Introduction

We consider finite, simple and undirected graphs $G = (V, E)$ with vertex set V and edge set E. A set of vertices $D \subseteq V$ of G is *dominating*, if every vertex in $V \setminus D$ has a neighbour in D. The minimum cardinality of a dominating set is the *domination number* $\gamma(G)$ of G. A set of vertices $I \subseteq V$ of G is *independent*, if no two vertices in I are adjacent. The maximum cardinality of an independent set is the *independence number* $\alpha(G)$ of G.

Dominating and independent sets are among the most well-studied graph sets. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [10, 11]. While much of the related research is devoted to $\gamma(G)$ and $\alpha(G)$, the problem of partitioning the vertex set into dominating sets [3, 7, 4] and even more the problem of partitioning the vertex set into independent sets, i.e. vertex colourings, have been extensively studied.

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Quite recently several authors have studied pairs of disjoint dominating sets. Kulli and Sigarkanti [14] introduced the *inverse domination number* $\gamma^{-1}(G)$ of a graph G as the minimum cardinality of a dominating set whose complement contains a minimum dominating set of G. Motivated by a false proof for the inequality $\gamma^{-1}(G) \leq \alpha(G)$ that appeared in [14], several authors [5, 8] studied this parameter. A classical result in domination theory due to Ore [15] is that if D is a minimal dominating set of a graph G with no isolated vertex, then $V \setminus D$ is also a dominating set of G. Thus every such graph G contains two disjoint dominating sets. In [13] Hedetniemi et al. initiate the study of the minimum cardinality $\gamma\gamma(G) = |D_1| + |D_2|$ of the union of two disjoint dominating sets D_1 and D_2 of a graph G with no isolated vertex. Similarly, they defined $\gamma i(G)$ as the minimum cardinality $|D_1|+|I_2|$ of the union of two disjoint dominating sets D_1 and I_2 of G for which I_2 is independent and they define $ii(G)$ as the minimum cardinality $|I_1| + |I_2|$ of the union of two disjoint independent dominating sets i_1 and I_2 of G. Various graph theoretic and algorithmic properties of these parameters are presented in [13].

For notation and graph theory terminology we in general follow [10]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. For a set S of vertices, the closed neighborhood of S is defined by $N_G[S] = \cup_{v \in S} N_G[v]$. If $X, Y \subseteq V$, then the set X is said to *dominate* the set Y if $Y \subseteq N_G[X]$. In particular if X dominates V, then X is a dominating set of G. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$.

2 Nine Problems posed in [13]

In this section, we list nine problems posed by Hedetniemi et al. in [13].

A) Characterize the graphs G for which $\gamma\gamma(G) = 2\gamma(G)$, i.e., characterize the graphs which have two disjoint minimum dominating sets. (Problem 1 in [13].)

B) Under what conditions does $ii(G)$ exist? (Problem 10 in [13].)

C) When is $\gamma\gamma(G) = \gamma i(G)$? (Problem 11 in [13].)

D) When is $\gamma i(G) = ii(G)$? (Problem 12 in [13].)

E) Is the calculation of $\gamma\gamma(G)$ NP-complete for bipartite graphs? (Problem 17 in [13].)

F) What is the complexity of the decision problem corresponding to $\gamma i(G)$? (Problem 13 in [13].)

G) For which class of trees T of order $n \geq 2$ is $\gamma\gamma(T) = 2(n + 1)/3$? (Problem 8 in [13]. Note that it is shown in [13] that $\gamma\gamma(T) \geq 2(n + 1)/3$ for all trees T of order $n \geq 2$.)

H) **Conjecture.** A tree T satisfies $\gamma\gamma(T) = 2\gamma(T)$ if and only if no vertex of T belongs to every minimum dominating set of T. (Problem 7 in [13].)

I) Does every tree of order $n \geq 2$ have a minimum dominating set whose complement contains an independent dominating set of T? (Problem 21 in [13].)

3 Results

Our aim in this paper is to solve the nine problems listed in Section 2.

3.1 Problem A

While trees with two disjoint minimum dominating sets were constructively characterized in [1] (cf. also [2, 6, 9, 12]), we give a somewhat negative 'solution' to Problem A by showing that the corresponding decision problem is NP-hard. We do not know whether this problem is actually in NP.

Theorem 1 *It is NP-hard to decide whether a given graph has two disjoint minimum dominating sets.*

Proof. Given a 3Sat instance $\mathcal{C}$ we will construct a graph G whose order is polynomially bounded in the size of $\mathcal{C}$ such that $\mathcal{C}$ is satisfiable if and only if G has two disjoint minimum dominating sets.

For every boolean variable x occurring in $\mathcal{C}$ we introduce a copy G_x of the gadget shown in the left part of Figure 1 which contains two specified vertices x and $\bar{x}$. Furthermore, for every clause C of $\mathcal{C}$ we introduce a copy G_C of the gadget shown in the right part of Figure 1 which contains one specified vertex C.

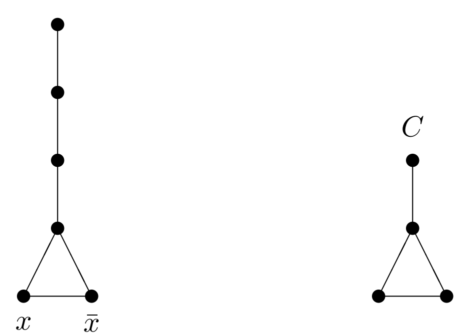

Figure 1. The gadgets G_x and G_C.

If the literal x occurs in clause C we connect the specified vertex x in G_x with the specified vertex C in G_C. (For an example see Figure 2 where $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee z \vee u\}$.) Let G denote the resulting graph.

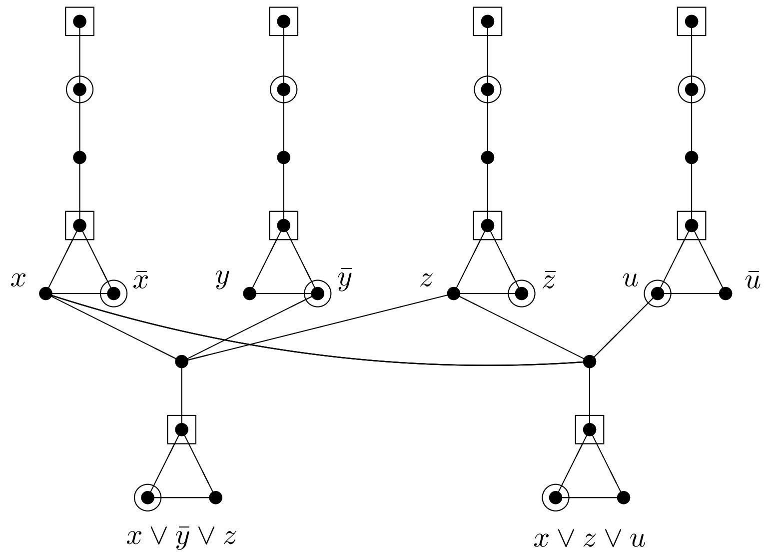

Figure 2. The graph G for $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee z \vee u\}$.

Let $\mathcal{C}$ use n boolean variables and contain m clauses. Note that the order of G is $6n+4m$. Every dominating set of G contains at least two vertices from every gadget G_x and at least one vertex from every gadget G_C. Conversely, choosing the two vertices at distance 1 and 3 from the endvertex in every gadget G_x and the dominating vertex in every gadget G_C yields a dominating set of G. This implies that $\gamma(G) = 2n + m$.

If $\mathcal{C}$ is satisfiable, then we consider a satisfying truth assignment for $\mathcal{C}$. The set of vertices corresponding to the true literals together with the neighbour of the endvertex in every gadget G_x and one of the two vertices of degree 2 in every gadget G_C yields a minimum dominating set D of G. Furthermore, choosing the two vertices at distance 0 and 3 from the endvertex in every gadget G_x and the dominating vertex in every gadget G_C yields a minimum dominating set of G which is disjoint from D.

Conversely, we assume now that G has two disjoint minimum dominating sets D_1 and D_2. By the above reasoning, each of D_1 and D_2 contains exactly one vertex from each gadget G_C. This implies that for every gadget G_C the specified vertex C must be dominated within one of D_1 and D_2 by a vertex not contained in G_C. Furthermore, for every gadget G_x the set $D_1 \cup D_2$ contains at most one of the two specified vertices x and $\bar{x}$. Therefore, the vertices in $D_1 \cup D_2$ corresponding to literals indicate a satisfying truth assignment for $\mathcal{C}$. Note that the truth value of a variable x for which neither x nor $\bar{x}$ is in $D_1 \cup D_2$ can be set arbitrarily. (The two minimum dominating sets indicated in Figure 2 correspond to setting x, y and z *false* and u *true*.) This completes the proof. $\square$

3.2 Problem B

As with Problem A, our 'solution' to Problem B is a hardness result.

Theorem 2 *It is NP-complete to decide whether a given graph has two disjoint independent dominating sets.*

Proof. The given decision problem is clearly in NP. Given a 3Sat instance $\mathcal{C}$ we will construct a graph G whose order is polynomially bounded in the size of $\mathcal{C}$ such that $\mathcal{C}$ is satisfiable if and only if G has two disjoint independent dominating sets.

For every boolean variable x occurring in $\mathcal{C}$ we introduce a copy G_x of the gadget shown in the left part of Figure 3 which contains two specified vertices x and $\bar{x}$. Furthermore, for every clause C of $\mathcal{C}$ we introduce a copy G_C of the gadget shown in the right part of Figure 3 which contains one specified vertex C.

Figure 3. The gadgets G_x and G_C.

If the literal x occurs in clause C we connect the specified vertex x in G_x with the specified vertex C in G_C. (For an example see Figure 4 where $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee z \vee u\}$.) Let G denote the resulting graph.

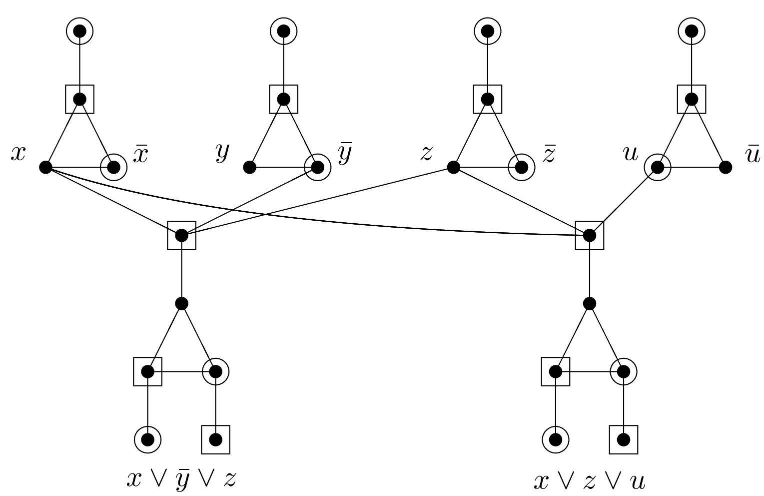

Figure 4. The graph G for $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee z \vee u\}$.

Let $\mathcal{C}$ use n boolean variables and contain m clauses. Note that the order of G is $4n+6m$.

If $\mathcal{C}$ is satisfiable, then we consider a satisfying truth assignment for $\mathcal{C}$. Choosing in every gadget G_x the endvertex and the vertex corresponding to the true literal and choosing in every gadget G_C an endvertex different from C and the neighbour of the other endvertex different from C yields an independent dominating set I of G. Furthermore, choosing in every gadget G_x the neighbour of the endvertex and choosing in every gadget G_C the vertex C and the two vertices not adjacent to C or contained in I yields an independent dominating set of G disjoint from I.

Conversely, we assume now that G has two disjoint independent dominating sets I_1 and I_2. Since in every gadget G_C the two vertices at distance two from C are necessarily in $I_1 \cup I_2$, the neighbour of C in G_C is not in $I_1 \cup I_2$. This implies that C is dominated within one of the two sets I_1 or I_2 by a vertex not contained in G_C. Clearly, at most one of the two vertices x and $\bar{x}$ in every gadget G_x can be in $I_1 \cup I_2$. Therefore, the vertices in $I_1 \cup I_2$ corresponding to literals indicate a satisfying truth assignment for $\mathcal{C}$. Again, the truth value of a variable x for which neither x nor $\bar{x}$ is in $I_1 \cup I_2$ can be set arbitrarily. (The two independent dominating sets indicated in Figure 4 correspond to setting x, y and z *false* and u *true*.) This completes the proof. $\square$

3.3 Problems C and D

As with Problems A and B, yet further hardness results.

Theorem 3 *Given a graph G the following two problems are NP-hard.*

(i) Decide whether G satisfies $\gamma\gamma(G) = \gamma i(G)$.

(ii) Decide whether G satisfies $\gamma i(G) = ii(G)$.

Proof. Given a 3Sat instance $\mathcal{C}$ we will construct two graphs G and G' whose order is polynomially bounded in the size of $\mathcal{C}$ such that $\mathcal{C}$ is satisfiable if and only if $\gamma\gamma(G) = \gamma i(G)$ if and only if $\gamma i(G') = ii(G')$.

For the construction of G we proceed as follows. For every boolean variable x occurring in $\mathcal{C}$ we introduce a copy G_x of the gadget shown in the left part of Figure 5 which contains two specified vertices x and $\bar{x}$. Furthermore, for every clause C of $\mathcal{C}$ we introduce a copy G_C of the gadget shown in the middle part of Figure 5 which contains one specified vertex C.

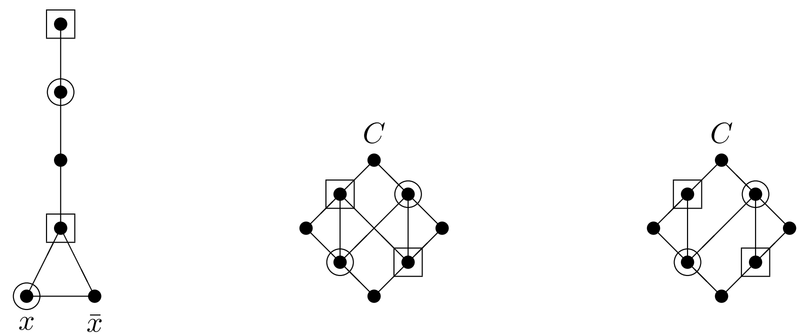

Figure 5. The gadgets G_x, G_C and G'_C.

If the literal x occurs in clause C we connect the specified vertex x in G_x with the specified vertex C in G_C. (For an example see Figure 6 where $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee z \vee u\}$.)

For the graph G' we proceed exactly as above using the gadget G'_C shown in the right part of Figure 5 instead of G_C.

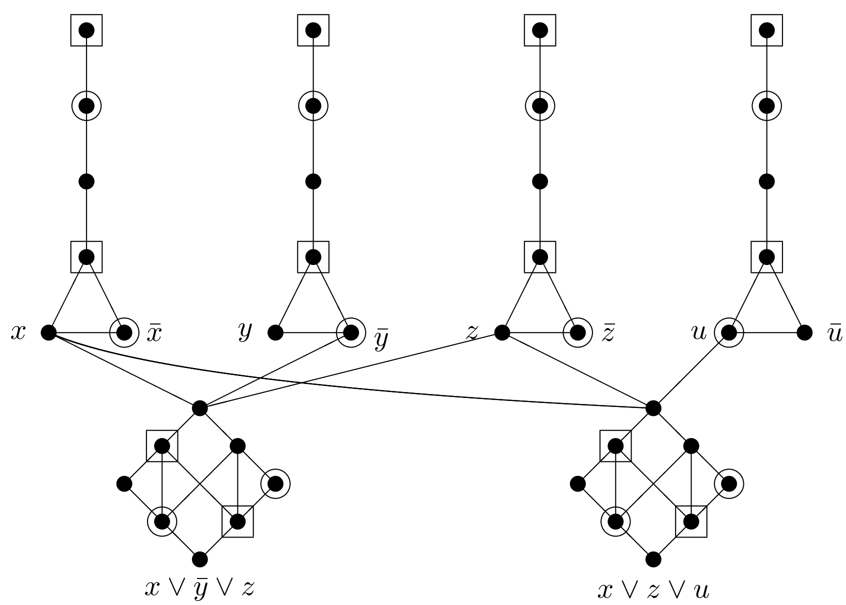

Figure 6. The graph G for $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee z \vee u\}$.

Let $\mathcal{C}$ use n boolean variables and contain m clauses. Note that the orders of G and G' are $6n + 8m$. Every dominating set of G contains at least two vertices from every gadget G_x and at least two vertices from every gadget G_C. Conversely, choosing in every gadget the vertices as indicated in Figure 5 yields two disjoint minimum dominating sets, i.e., $\gamma\gamma(G) = 2\gamma(G) = 4n + 4m$. Similarly, $\gamma i(G') = 2\gamma(G') = 4n + 4m$.

If $\mathcal{C}$ is satisfiable, then we consider a satisfying truth assignment for $\mathcal{C}$. We choose the two disjoint minimum dominating sets described above such that from every gadget G_x the vertex corresponding to the true literal is in one of the two sets. Furthermore, in every gadget G_C we choose vertices as indicated in Figure 6. This yields two disjoint minimum dominating sets one of which is independent, i.e., $\gamma\gamma(G) = \gamma i(G)$. Similar arguments yield $\gamma i(G') = ii(G')$.

Conversely, we assume now that G satisfies $\gamma\gamma(G) = \gamma i(G)$. Let D_1 and I_2 be two disjoint dominating sets such that I_2 is independent and $|D_1| + |I_2| = \gamma\gamma(G) = \gamma i(G) = 2\gamma(G)$, i.e., D_1 and I_2 are both minimum dominating. By the above reasoning, each of D_1 and I_2 contains exactly two vertices from each gadget G_C. This easily implies that in every gadget G_C the specified vertex C is dominated within one of D_1 and I_2 by a vertex not contained in G_C. Furthermore, for every gadget G_x the set $D_1 \cup I_2$ contains at most one of the two specified vertices x and $\bar{x}$. Therefore, the vertices in $D_1 \cup I_2$ corresponding to literals indicate a satisfying truth assignment for $\mathcal{C}$. (The two minimum dominating sets indicated in Figure 6 correspond to setting x, y and z *false* and u *true*.) Again, if we assume that G' satisfies $\gamma i(G') = ii(G')$, then the same train of thought implies that $\mathcal{C}$ is satisfiable. This completes the proof. $\square$

3.4 Problems E and F

In [13] it is shown that the calculation of $\gamma\gamma(G)$ is NP-hard even when restricted to chordal graphs. In Problem E, the authors in [13] ask about the complexity for the class of bipartite graphs, while in Problem F they ask about the complexity of the decision problem corresponding to $\gamma i(G)$. We prove that the corresponding decision problems are NP-complete. Note that Theorem 2 and the statement made about $ii(G)$ in Theorem 4 that follows do not imply each other.

Theorem 4 *Given a bipartite graph G and given an integer k the following three problems are NP-complete.*

(i) *Decide whether G has two disjoint dominating sets D_1 and D_2 with $|D_1| + |D_2| \leq k$.*

(ii) *Decide whether G has two disjoint dominating sets D_1 and D_2 with $|D_1| + |D_2| \leq k$ such that D_2 is independent.*

(iii) *Decide whether G has two disjoint independent dominating sets D_1 and D_2 with $|D_1| + |D_2| \leq k$.*

Proof. The three decision problems are clearly in NP. Given a 3Sat instance $\mathcal{C}$ we will construct a graph G whose order is polynomially bounded in the size of $\mathcal{C}$ and specify an integer k also polynomially bounded in the size of $\mathcal{C}$ such that if $\mathcal{C}$ is satisfiable, then $ii(G) \leq k$ and if $\gamma\gamma(G) \leq k$, then $\mathcal{C}$ is satisfiable. This clearly implies the desired statements.

For every boolean variable x occurring in $\mathcal{C}$ we introduce a copy G_x of the gadget shown in the left part of Figure 7 which contains two specified vertices x and $\bar{x}$. Furthermore, for every clause C of $\mathcal{C}$ we introduce a copy G_C of the gadget shown in the right part of Figure 7 which contains two specified vertices C and $\bar{C}$.

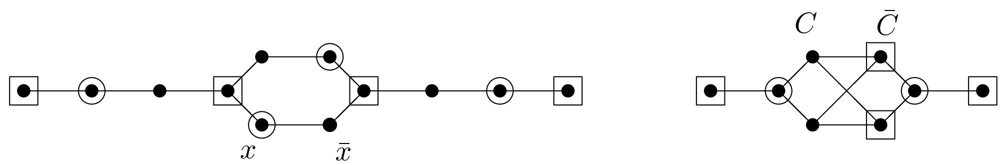

Figure 7. The gadgets G_x and G_C.

If the (unnegated) variable x occurs in clause C we connect the specified vertex x in G_x with the specified vertex C in G_C. Similarly, if the negated variable $\bar{x}$ occurs in clause C we connect the specified vertex $\bar{x}$ in G_x with the specified vertex $\bar{C}$ in G_C. Note that this way of adding edges to the disjoint union of the bipartite gadgets results in a bipartite graph. (For an example see Figure 8 where $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee z \vee u\}$.) Let G denote the resulting graph.

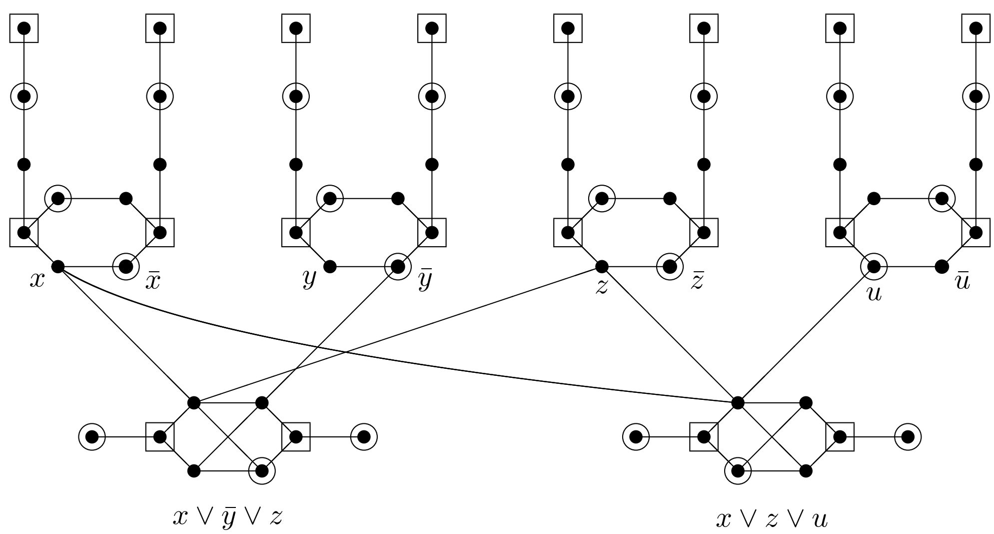

Figure 8. The graph G for $\mathcal{C} = \{x \vee \bar{y} \vee z, x \vee z \vee u\}$.

Let $\mathcal{C}$ use n boolean variables and contain m clauses. Note that the order of G is $12n+8m$. Let $k = 8n + 5m$.

First we assume that $\mathcal{C}$ is satisfiable and describe how to obtain two disjoint dominating sets D_1 and D_2 of G with $|D_1|+|D_2| \leq k$. Consider a satisfying truth assignment for $\mathcal{C}$. We choose in every gadget G_x the vertices for the sets D_1 and D_2 as indicated in the left part of Figure 7 or its mirror image such that D_1 contains the vertex corresponding to the true literal among x or $\bar{x}$. Since the truth assignment is satisfying, at least one of the vertices C or $\bar{C}$ in every gadget G_C is dominated in D_1 by a vertex not contained in $V(G_C)$. This implies that the two sets D_1 and D_2 can be extended as indicated in Figure 8 using a total of five vertices in each of the gadgets G_C. Hence, $|D_1| + |D_2| = k$.

Next, we assume that G has two disjoint dominating sets D_1 and D_2 such that $|D_1| + |D_2| \leq k$. In every gadget G_x, the set $V(G_x) \cap (D_1 \cup D_2)$ contains at least eight vertices in order to dominate the ten vertices on the path $G_x - \{x, \bar{x}\}$. Furthermore, if $V(G_x) \cap (D_1 \cup D_2)$ contains exactly eight vertices, then at least one of x and $\bar{x}$ is not contained in $D_1 \cup D_2$.

If for some gadget G_C neither C nor $\bar{C}$ are dominated by a vertex in $D_1 \cup D_2$ not contained in $V(G_C)$, then $V(G_C) \cap (D_1 \cup D_2)$ contains at least six vertices. (One possible configuration is shown in the right part of Figure 7.) Furthermore, if for some gadget G_C one or both of C and $\bar{C}$ are dominated by vertices in $D_1 \cup D_2$ not contained in $V(G_C)$, then $V(G_C) \cap (D_1 \cup D_2)$ contains at least five vertices.

Since $|D_1| + |D_2| \leq 8n + 5m$, we obtain that for every gadget G_x at most one of x and $\bar{x}$ is contained in $D_1 \cup D_2$ and for every gadget G_C one of C and $\bar{C}$ is dominated by a vertex in $D_1 \cup D_2$ not contained in $V(G_C)$. This implies that the vertices contained in $D_1 \cup D_2$ corresponding to literals indicate a satisfying truth assignment for $\mathcal{C}$ and the proof is complete. $\Box$

3.5 Problem G

As remark earlier, it is shown in [13] that $\gamma\gamma(T) \geq 2(n+1)/3$ for all trees T of order $n \geq 2$. In Problem G, the authors ask for a characterization of the trees achieving equality in this bound.

Theorem 5 *If $T = (V, E)$ is a tree of order n, then $\gamma\gamma(T) \geq 2(n+1)/3$ with equality if and only if V can be partitioned into two sets D and R such that D induces a perfect matching and R is an independent set all vertices of which have degree 2 in T.*

Proof. Let T be a tree of order n and let D_1 and D_2 be two disjoint dominating sets of T such that $\gamma\gamma(T) = |D_1| + |D_2|$. We assume that $|D_1| \geq |D_2|$. Let $D = D_1 \cup D_2$ and let $R = V \setminus D$. Since every vertex in R has a neighbour in D_1 and a neighbour in D_2 and every vertex in D_1 has a neighbour in D_2, counting the edges of T yields

$$n - 1 \geq 2|R| + |D_1| \geq 2|R| + |D|/2 = 2(n - \gamma\gamma(T)) + \gamma\gamma(T)/2,$$

which implies $\gamma\gamma(T) \geq 2(n+1)/3$.

If $\gamma\gamma(T) = 2(n+1)/3$, then equality holds throughout the above inequality chain. This implies that $|D_1| = |D_2|$, every vertex in R has exactly one neighbour in D_1 and one neighbour in D_2, every vertex from D_1 has exactly one neighbour in D_2 and the three sets D_1, D_2 and R are independent. Since every vertex of D_2 has at least one neighbour in D_1, the set D induces a perfect matching and the structure of T is as described in the statement of the result.

Conversely, we assume now that V can be partitioned into two sets D and R such that D induces a perfect matching and R is an independent set all vertices of which have degree 2 in T. We will prove by induction on the order n of T that $\gamma\gamma(T) = 2(n+1)/3$. More specifically, we prove that D can be partitioned into two independents sets D_1 and D_2 which are both dominating. Note that, by the assumptions, such sets D_1 and D_2 satisfy $|D_1| + |D_2| = 2(n+1)/3$. If $n = 2$, then the statement is trivial. Hence, we may assume that $n \geq 3$. Let uv be an edge which corresponds to an endvertex of the tree which arises from T by contracting all edges of the perfect matching induced by D. Note that after these contractions all vertices in R are still of degree 2. This implies that we may assume that u is an endvertex of T and v has degree 2 in T. Let w be the neighbour of v different from u. Clearly, $w \in R$. The vertex set $V \setminus \{u,v,w\}$ of the tree $T' = T - \{u,v,w\}$ can be partitioned into two sets $D' = D \setminus \{u,v\}$ and $R' = R \setminus \{w\}$ such that D' induces a perfect matching and R' is an independent set all vertices of which have degree 2 in T'. Hence, by induction, D' can be partitioned into two independent sets D'_1 and D'_2 both of which are dominating in T'. We may assume that the neighbour of w different from v belongs to D'_1. Now the two sets $D_1 = D'_1 \cup \{u\}$ and $D_2 = D'_2 \cup \{v\}$ are independent and dominating in T and partition D which completes the proof. □

3.6 Problem H

In Problem H the authors conjecture that for a tree T the equality $\gamma\gamma(T) = 2\gamma(T)$ is equivalent to the property that no vertex of T belongs to every minimum dominating set of T. While this property is obviously necessary, we describe an example disproving the conjecture.

Observation 6 *There are trees T for which no vertex belongs to every minimum dominating set of T and which do not have two disjoint minimum dominating sets, i.e., $\gamma\gamma(T) > 2\gamma(T)$.*

Proof. The tree two copies of which are shown in Figure 9 has domination number 7 and the two indicated minimum dominating sets show that no vertex belongs to every minimum dominating set of T. On the other hand it is easy to see that the union of every two disjoint dominating sets of T contains at least five vertices in each of the indicated rectangular boxes which implies that one of the sets cannot be minimum. □

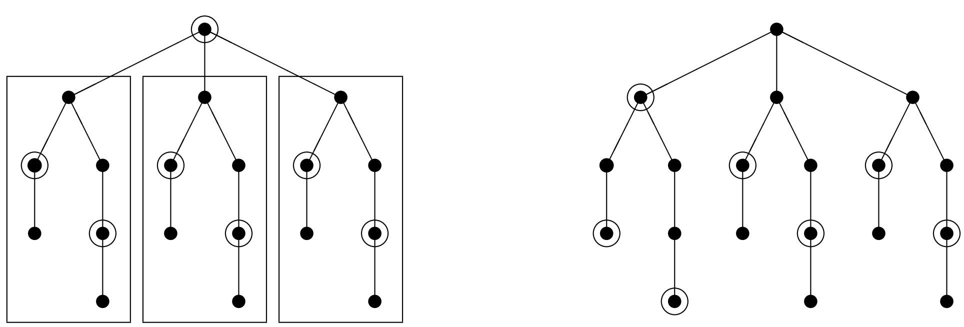

Figure 9. A counterexample to the conjecture posed in Problem H.

3.7 Problem I

In Problem I, it is asked whether every tree of order n has a minimum dominating set whose complement contains an independent dominating set. We answer this question in the affirmative. For this purpose, given a rooted tree T, a set D of vertices of T and a vertex $v \in D$, we define an *external D-private child of v in T* to be a child of v in $N_T(v) \setminus N_T[D \setminus \{v\}]$. Hence if u is an external D-private child of v in T, then $u \notin D$, u is a child of v in T, and $N_T(u) \cap D = \{v\}$.

Theorem 7 *Every tree of order at least two has a minimum dominating set and an independent dominating set which are disjoint.*

Proof. Let u be an endvertex of T. Let D be a minimum dominating set containing a neighbour r of u such that

$$f(D) := \sum_{v \in D} \mathrm{dist}_T(v, r)$$

is minimum. Root T at r. Note that u is an external D-private child of r in T. If some vertex $v \in D \setminus \{r\}$ has no external D-private child in T, then the parent w of v is not in D. Now the set $D' = (D \setminus \{v\}) \cup \{w\}$ is a minimum dominating set of T containing r with $f(D') = f(D) - 1$, which is a contradiction. Hence all vertices in D have external D-private children in T. Clearly, a set I containing exactly one external D-private child of every vertex in D is an independent set and a maximal independent subset of $V \setminus D$ which contains I is a dominating set of T. This completes the proof. □

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